CHAPTER 4

PROPERTIES OF MEMBERSHIP FUNCTIONS, FUZZIFICATION, AND DEFUZZIFICATION

''Let's consider your age, to begin with – how old are you?'' ''I'm seven and a half, exactly.'' ''You needn't say 'exactually,' '' the Queen remarked; ''I can believe it without that. Now I'll give you something to believe. I'm just one hundred and one, five months, and a day.'' ''I can't believe that!'' said Alice.
''Can't you?'' the Queen said in a pitying tone. ''Try again; draw a long breath, and shut your eyes.'' Alice laughed. ''There's no use trying,'' she said; ''one can't believe impossible things.''

Lewis Carroll
Through the Looking Glass, 1871

It is one thing to compute, to reason, and to model with fuzzy information; it is another to apply the fuzzy results to the world around us. Despite the fact that the bulk of the information we assimilate every day is fuzzy, like the age of people in the Lewis Carroll example above, most of the actions or decisions implemented by humans or machines are crisp or binary. The decisions we make that require an action are binary, the hardware we use is binary, and certainly the computers we use are based on binary digital instructions. For example, in making a decision about developing a new engineering product the eventual decision is to go forward with development or not; the fuzzy choice to ''maybe go forward'' might be acceptable in planning stages, but eventually funds are released for development or they are not. In giving instructions to an aircraft autopilot, it is not possible to turn the plane ''slightly to the west''; an autopilot device does not understand the natural language

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of a human. We have to turn the plane by 15°, for example, a crisp number. An electrical circuit typically is either on or off, not partially on.

The bulk of this textbook illustrates procedures to ''fuzzify'' the mathematical and engineering principles we have so long considered to be deterministic. But in various applications and engineering scenarios there will be a need to ''defuzzify'' the fuzzy results we generate through a fuzzy systems analysis. In other words, we may eventually find a need to convert the fuzzy results to crisp results. For example, in classification and pattern recognition (see Chapter 11) we may want to transform a fuzzy partition or pattern into a crisp partition or pattern; in control (see Chapter 13) we may want to give a single-valued input to a semiconductor device instead of a fuzzy input command. This ''defuzzification'' has the result of reducing a fuzzy set to a crisp single-valued quantity, or to a crisp set; of converting a fuzzy matrix to a crisp matrix; or of making a fuzzy number a crisp number.

Mathematically, the defuzzification of a fuzzy set is the process of ''rounding it off'' from its location in the unit hypercube to the nearest (in a geometric sense) vertex (see Chapter 1). If one thinks of a fuzzy set as a collection of membership values, or a vector of values on the unit interval, defuzzification reduces this vector to a single scalar quantity – presumably to the most typical (prototype) or representative value. Various popular forms of converting fuzzy sets to crisp sets or to single scalar values are introduced later in this chapter.

FEATURES OF THE MEMBERSHIP FUNCTION

Since all information contained in a fuzzy set is described by its membership function, it is useful to develop a lexicon of terms to describe various special features of this function. For purposes of simplicity, the functions shown in the following figures will all be continuous, but the terms apply equally for both discrete and continuous fuzzy sets. Figure 4.1 assists in this description.

The *core* of a membership function for some fuzzy set $\underset{\sim}{A}$ is defined as that region of the universe that is characterized by complete and full membership in the set $\underset{\sim}{A}$. That is, the core comprises those elements x of the universe such that $\mu_{\underset{\sim}{A}}(x) = 1$.

The *support* of a membership function for some fuzzy set $\underset{\sim}{A}$ is defined as that region of the universe that is characterized by nonzero membership in the set $\underset{\sim}{A}$. That is, the support comprises those elements x of the universe such that $\mu_{\underset{\sim}{A}}(x) > 0$.

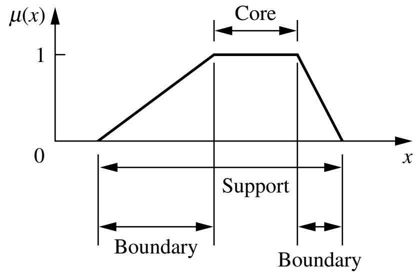

FIGURE 4.1
Core, support, and boundaries of a fuzzy set.

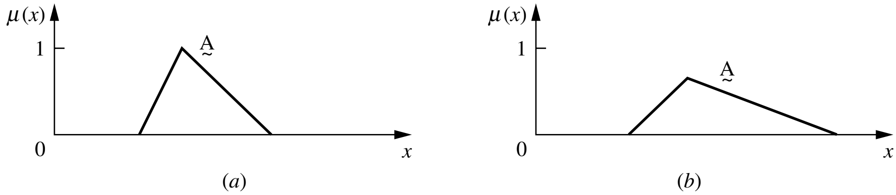

FIGURE 4.2
Fuzzy sets that are normal (*a*) and subnormal (*b*).

The *boundaries* of a membership function for some fuzzy set $\underset{\sim}{A}$ are defined as that region of the universe containing elements that have a nonzero membership but not complete membership. That is, the boundaries comprise those elements x of the universe such that $0 < \mu_{\underset{\sim}{A}}(x) < 1$. These elements of the universe are those with some *degree* of fuzziness, or only partial membership in the fuzzy set $\underset{\sim}{A}$. Figure 4.1 illustrates the regions in the universe comprising the core, support, and boundaries of a typical fuzzy set.

A *normal* fuzzy set is one whose membership function has at least one element x in the universe whose membership value is unity. For fuzzy sets where one and only one element has a membership equal to one, this element is typically referred to as the *prototype* of the set, or the prototypical element. Figure 4.2 illustrates typical normal and subnormal fuzzy sets.

A *convex* fuzzy set is described by a membership function whose membership values are strictly monotonically increasing, or whose membership values are strictly monotonically decreasing, or whose membership values are strictly monotonically increasing then strictly monotonically decreasing with increasing values for elements in the universe. Said another way, if, for any elements x, y, and z in a fuzzy set $\underset{\sim}{A}$, the relation $x < y < z$ implies that

$$\mu_{\underset{\sim}{A}}(y) \geq \min[\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{A}}(z)] \tag{4.1}$$

then $\underset{\sim}{A}$ is said to be a convex fuzzy set [Ross, 1995]. Figure 4.3 shows a typical convex fuzzy set and a typical nonconvex fuzzy set. It is important to remark here that this definition of convexity is *different* from some definitions of the same term in mathematics. In some areas of mathematics, convexity of shape has to do with whether a straight line through any part of the shape goes outside the boundaries of that shape. This definition of convexity is *not* used here; Eq. (4.1) succinctly summarizes our definition of convexity.

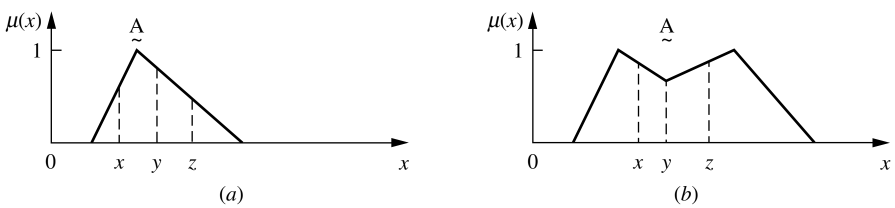

FIGURE 4.3
Convex, normal fuzzy set (*a*) and nonconvex, normal fuzzy set (*b*).

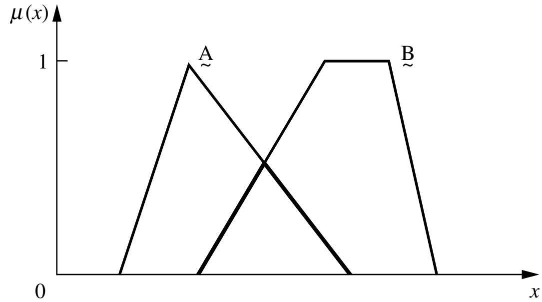

FIGURE 4.4
The intersection of two convex fuzzy sets produces a convex fuzzy set.

A special property of two convex fuzzy sets, say $\underset{\sim}{A}$ and $\underset{\sim}{B}$, is that the intersection of these two convex fuzzy sets is also a convex fuzzy set, as shown in Fig. 4.4. That is, for $\underset{\sim}{A}$ and $\underset{\sim}{B}$, which are both convex, $\underset{\sim}{A} \cap \underset{\sim}{B}$ is also convex.

The *crossover points* of a membership function are defined as the elements in the universe for which a particular fuzzy set $\underset{\sim}{A}$ has values equal to 0.5, i.e., for which $\mu_{\underset{\sim}{A}}(x) = 0.5$.

The *height* of a fuzzy set $\underset{\sim}{A}$ is the maximum value of the membership function, i.e., $\text{hgt}(\underset{\sim}{A}) = \max\{\mu_{\underset{\sim}{A}}(x)\}$. If the $\text{hgt}(\underset{\sim}{A}) < 1$, the fuzzy set is said to be subnormal. The $\text{hgt}(\underset{\sim}{A})$ may be viewed as the degree of validity or credibility of information expressed by $\underset{\sim}{A}$ [Klir and Yuan, 1995].

If $\underset{\sim}{A}$ is a convex single-point normal fuzzy set defined on the real line, then $\underset{\sim}{A}$ is often termed a *fuzzy number*.

VARIOUS FORMS

The most common forms of membership functions are those that are normal and convex. However, many operations on fuzzy sets, hence operations on membership functions, result in fuzzy sets that are subnormal and nonconvex. For example, the extension principle to be discussed in Chapter 12 and the union operator both can produce subnormal or nonconvex fuzzy sets.

Membership functions can be symmetrical or asymmetrical. They are typically defined on one-dimensional universes, but they certainly can be described on multidimensional (or n-dimensional) universes. For example, the membership functions shown in this chapter are one-dimensional curves. In two dimensions these curves become surfaces and for three or more dimensions these surfaces become hypersurfaces. These hypersurfaces, or curves, are simple mappings from combinations of the parameters in the n-dimensional space to a membership value on the interval [0, 1]. Again, this membership value expresses the degree of membership that the specific combination of parameters in the n-dimensional space has in a particular fuzzy set defined on the n-dimensional universe of discourse. The hypersurfaces for an n-dimensional universe are analogous to joint probability density functions; but, of course, the mapping for the membership function is to membership in a particular set and not to relative frequencies, as it is for probability density functions.

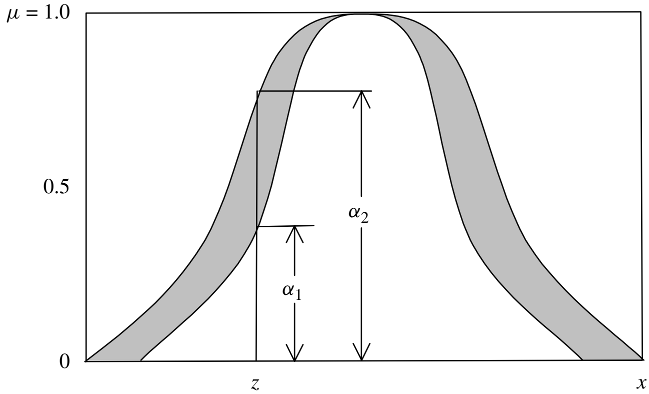

FIGURE 4.5
An interval-valued membership function.

Fuzzy sets of the types depicted in Fig. 4.2 are by far the most common ones encountered in practice; they are described by *ordinary membership functions*. However, several other types of fuzzy membership functions have been proposed [Klir and Yuan, 1995] as *generalized membership functions*. The primary reason for considering other types of membership functions is that the values used in developing ordinary membership functions are often overly precise. They require that each element of the universe x on which the fuzzy set $\underset{\sim}{A}$ is defined be assigned a specific membership value, $\mu_{\underset{\sim}{A}}(x)$. Suppose the level of information is not adequate to specify membership functions with this precision. For example, we may only know the upper and lower bounds of membership grades for each element of the universe for a fuzzy set. Such a fuzzy set would be described by an *interval-valued membership function*, such as the one shown in Fig. 4.5. In this figure, for a particular element, $x = z$, the membership in a fuzzy set $\underset{\sim}{A}$, i.e., $\mu_{\underset{\sim}{A}}(z)$, would be expressed by the membership interval $[\alpha_1, \alpha_2]$. Interval-valued fuzzy sets can be generalized further by allowing their intervals to become fuzzy. Each membership interval then becomes an ordinary fuzzy set. This type of membership function is referred to in the literature as a *type-2 fuzzy set*. Other generalizations of the fuzzy membership functions are available as well [see Klir and Yuan, 1995].

FUZZIFICATION

Fuzzification is the process of making a crisp quantity fuzzy. We do this by simply recognizing that many of the quantities that we consider to be crisp and deterministic are actually not deterministic at all: They carry considerable uncertainty. If the form of uncertainty happens to arise because of imprecision, ambiguity, or vagueness, then the variable is probably fuzzy and can be represented by a membership function.

In the real world, hardware such as a digital voltmeter generates crisp data, but these data are subject to experimental error. The information shown in Fig. 4.6 shows one possible range of errors for a typical voltage reading and the associated membership function that might represent such imprecision.

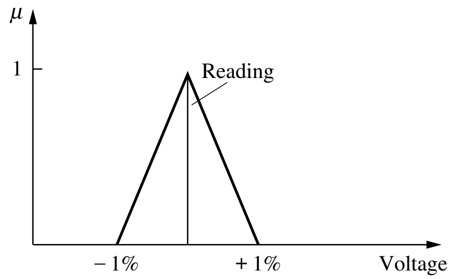

FIGURE 4.6
Membership function representing imprecision in ''crisp voltage reading.''

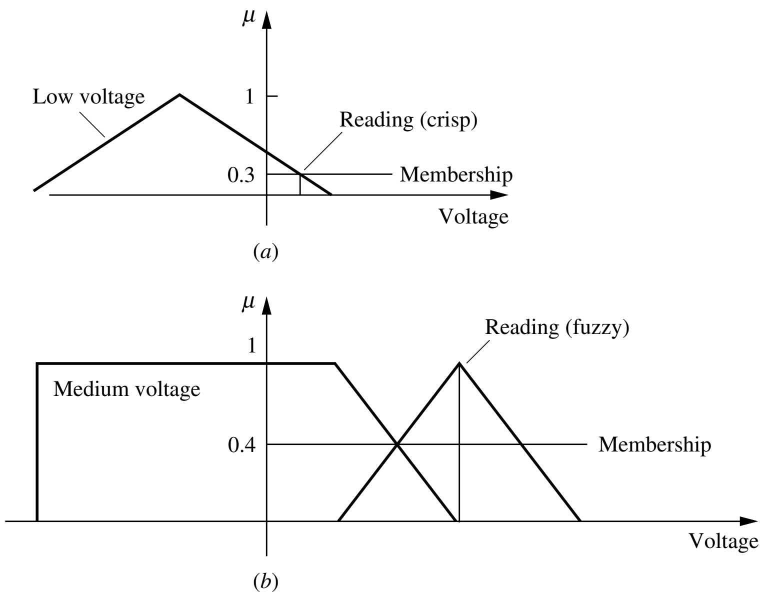

FIGURE 4.7
Comparisons of fuzzy sets and crisp or fuzzy readings: (*a*) fuzzy set and crisp reading; (*b*) fuzzy set and fuzzy reading.

The representation of imprecise data as fuzzy sets is a useful but not mandatory step when those data are used in fuzzy systems. This idea is shown in Fig. 4.7, where we consider the data as a crisp reading, Fig. 4.7*a*, or as a fuzzy reading, as shown in Fig. 4.7*b*. In Fig. 4.7*a* we might want to compare a crisp voltage reading to a fuzzy set, say ''low voltage.'' In the figure we see that the crisp reading intersects the fuzzy set ''low voltage'' at a membership of 0.3, i.e., the fuzzy set and the reading can be said to agree at a membership value of 0.3. In Fig. 4.7*b* the intersection of the fuzzy set ''medium voltage'' and a fuzzified voltage reading occurs at a membership of 0.4. We can see in Fig. 4.7*b* that the set intersection of the two fuzzy sets is a small triangle, whose largest membership occurs at the membership value of 0.4.

We will say more about the importance of fuzzification of crisp variables in Chapters 8 and 13 of this text. In Chapter 8 the topic is simulation, and the inputs for any nonlinear or complex simulation will be expressed as fuzzy sets. If the process is inherently quantitative or the inputs derive from sensor measurements, then these crisp numerical inputs could be fuzzified in order for them to be used in a fuzzy inference system (to be discussed in Chapter 5). In Chapter 13 the topic is fuzzy control, and, again, this is a discipline where the inputs generally originate from a piece of hardware, or a sensor and the measured input could be fuzzified for utility in the rule-based system which describes the fuzzy controller. If the system to be controlled is not hardware based, e.g., the control of an economic system or the control of an ecosystem subjected to a toxic chemical, then the inputs could be scalar quantities arising from statistical sampling, or other derived numerical quantities. Again, for utility in fuzzy systems, these scalar quantities could first be fuzzified, i.e., translated into a membership function, and then used to form the input structure necessary for a fuzzy system.

DEFUZZIFICATION TO CRISP SETS

We begin by considering a fuzzy set $\underset{\sim}{\mathrm{A}}$, then define a lambda-cut set, A_λ, where $0 \leq \lambda \leq 1$. The set A_λ is a crisp set called the lambda (λ)-cut (or alpha-cut) set of the fuzzy set $\underset{\sim}{\mathrm{A}}$, where $\mathrm{A}_\lambda = \{x|\mu_{\underset{\sim}{\mathrm{A}}}(x) \geq \lambda\}$. Note that the λ-cut set A_λ does not have a tilde underscore; it is a crisp set derived from its parent fuzzy set, $\underset{\sim}{\mathrm{A}}$. Any particular fuzzy set $\underset{\sim}{\mathrm{A}}$ can be transformed into an infinite number of λ-cut sets, because there are an infinite number of values λ on the interval [0, 1].

Any element $x \in \mathrm{A}_\lambda$ belongs to $\underset{\sim}{\mathrm{A}}$ with a grade of membership that is greater than or equal to the value λ. The following example illustrates this idea.

Example 4.1. Let us consider the discrete fuzzy set, using Zadeh's notation, defined on universe $\mathrm{X} = \{a, b, c, d, e, f\}$,

$$\underset{\sim}{\mathrm{A}} = \left\{\frac{1}{a} + \frac{0.9}{b} + \frac{0.6}{c} + \frac{0.3}{d} + \frac{0.01}{e} + \frac{0}{f}\right\}$$

This fuzzy set is shown schematically in Fig. 4.8. We can reduce this fuzzy set into several λ-cut sets, all of which are crisp. For example, we can define λ-cut sets for the values of $\lambda = 1$, 0.9, 0.6, 0.3, 0^+, and 0.

$$\mathrm{A}_1 = \{a\}, \quad \mathrm{A}_{0.9} = \{a, b\}$$
$$\mathrm{A}_{0.6} = \{a, b, c\}, \quad \mathrm{A}_{0.3} = \{a, b, c, d\}$$
$$\mathrm{A}_{0^+} = \{a, b, c, d, e\}, \quad \mathrm{A}_0 = \mathrm{X}$$

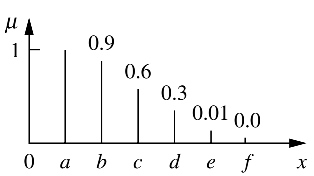

FIGURE 4.8
A discrete fuzzy set $\underset{\sim}{\mathrm{A}}$.

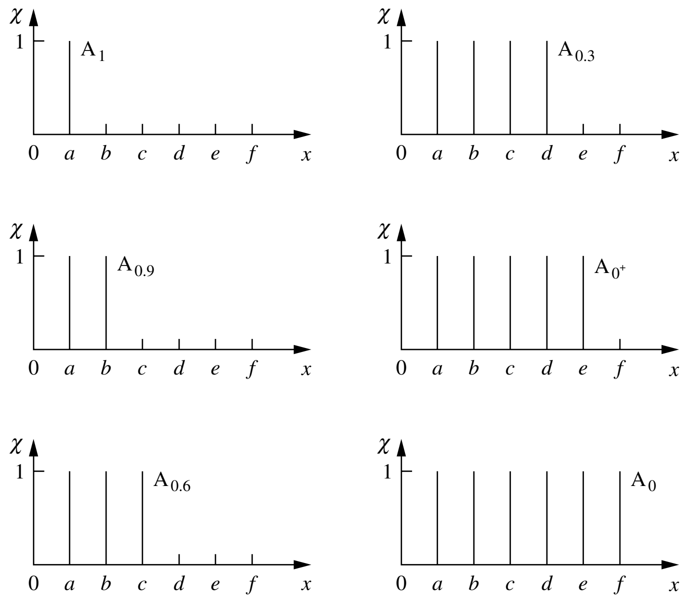

FIGURE 4.9
Lambda-cut sets for $\lambda = 1, 0.9, 0.6, 0.3, 0^+, 0$.

The quantity $\lambda = 0^+$ is defined as a small ''ε'' value >0, i.e., a value just greater than zero. By definition, $\lambda = 0$ produces the universe X, since all elements in the universe have at least a 0 membership value in any set on the universe. Since all A_λ are crisp sets, all the elements just shown in the example λ-cut sets have unit membership in the particular λ-cut set. For example, for $\lambda = 0.3$, the elements a, b, c, and d of the universe have a membership of 1 in the λ-cut set, $A_{0.3}$, and the elements e and f of the universe have a membership of 0 in the λ-cut set, $A_{0.3}$. Figure 4.9 shows schematically the crisp λ-cut sets for the values $\lambda = 1$, 0.9, 0.6, 0.3, 0^+, and 0. Notice in these plots of membership value versus the universe X that the effect of a λ-cut is to rescale the membership values: to one for all elements of the fuzzy set $\underset{\sim}{A}$ having membership values greater than or equal to λ, and to zero for all elements of the fuzzy set $\underset{\sim}{A}$ having membership values less than λ.

We can express λ-cut sets using Zadeh's notation. For the example, λ-cut sets for the values $\lambda = 0.9$ and 0.25 are given here:

$$A_{0.9} = \left\{\frac{1}{a} + \frac{1}{b} + \frac{0}{c} + \frac{0}{d} + \frac{0}{e} + \frac{0}{f}\right\} \qquad A_{0.25} = \left\{\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{0}{e} + \frac{0}{f}\right\}$$

λ-cut sets obey the following four very special properties:

1. $(\underset{\sim}{A} \cup \underset{\sim}{B})_\lambda = A_\lambda \cup B_\lambda$ $\quad (4.1a)$

2. $(\underset{\sim}{A} \cap \underset{\sim}{B})_\lambda = A_\lambda \cap B_\lambda$ $\quad (4.1b)$

3. $(\overline{\underset{\sim}{A}})_\lambda \neq \overline{A}_\lambda$ except for a value of $\lambda = 0.5$ $\quad (4.1c)$

4. For any $\lambda \le \alpha$, where $0 \le \alpha \le 1$, it is true that $A_\alpha \subseteq A_\lambda$, where $A_0 = X$ $\quad (4.1d)$

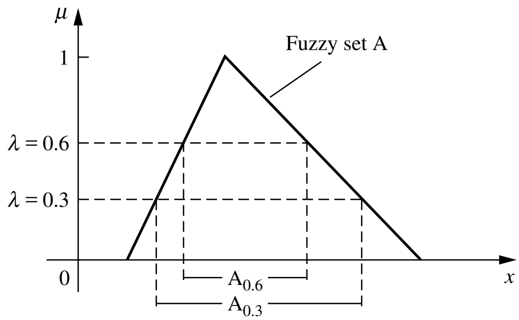

FIGURE 4.10
Two different λ-cut sets for a continuous-valued fuzzy set.

These properties show that λ-cuts on standard operations on fuzzy sets are equivalent with standard set operations on λ-cut sets. The last operation, Eq. (4.1d), can be shown more conveniently using graphics. Figure 4.10 shows a continuous-valued fuzzy set with two λ-cut values. Notice in the graphic that for $\lambda = 0.3$ and $\alpha = 0.6$, $A_{0.3}$ has a greater domain than $A_{0.6}$, i.e., for $\lambda \leq \alpha$ $(0.3 \leq 0.6)$, $A_{0.6} \subseteq A_{0.3}$.

In this chapter, various definitions of a membership function are discussed and illustrated. Many of these same definitions arise through the use of λ-cut sets. As seen in Fig. 4.1, we can provide the following definitions for a convex fuzzy set $\underset{\sim}{A}$. The core of $\underset{\sim}{A}$ is the $\lambda = 1$ cut set, A_1. The support of $\underset{\sim}{A}$ is the λ-cut set A_{0^+}, where $\lambda = 0^+$, or symbolically, $A_{0^+} = \{x \mid \mu_{\underset{\sim}{A}(x)} > 0\}$. The intervals $[A_{0^+}, A_1]$ form the boundaries of the fuzzy set $\underset{\sim}{A}$, i.e., those regions that have membership values between 0 and 1 (exclusive of 0 and 1): that is, for $0 < \lambda < 1$.

λ-CUTS FOR FUZZY RELATIONS

In Chapter 3, a biotechnology example, Example 3.11, was developed using a fuzzy relation that was reflexive and symmetric. Recall this matrix,

$$\underset{\sim}{R} = \begin{bmatrix} 1 & 0.8 & 0 & 0.1 & 0.2 \\ 0.8 & 1 & 0.4 & 0 & 0.9 \\ 0 & 0.4 & 1 & 0 & 0 \\ 0.1 & 0 & 0 & 1 & 0.5 \\ 0.2 & 0.9 & 0 & 0.5 & 1 \end{bmatrix}$$

We can define a λ-cut procedure for relations similar to the one developed for sets. Consider a fuzzy relation $\underset{\sim}{R}$, where each row of the relational matrix is considered a fuzzy set, i.e., the jth row in $\underset{\sim}{R}$ represents a discrete membership function for a fuzzy set, $\underset{\sim}{R}_j$. Hence, a fuzzy relation can be converted to a crisp relation in the following manner. Let us define $R_\lambda = \{(x, y) \mid \mu_{\underset{\sim}{R}}(x, y) \geq \lambda\}$ as a λ-cut relation of the fuzzy relation, $\underset{\sim}{R}$. Since in this case $\underset{\sim}{R}$ is a two-dimensional array defined on the universes X and Y, then any pair $(x, y) \in R_\lambda$ belongs to $\underset{\sim}{R}$ with a ‘‘strength’’ of relation greater than or equal to λ. These ideas for relations can be illustrated with an example.

Example 4.2. Suppose we take the fuzzy relation from the biotechnology example in Chapter 3 (Example 3.11), and perform λ-cut operations for the values of $\lambda = 1, 0.9, 0$. These crisp relations are given below:

$$\lambda = 1,\ \mathrm{R}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\lambda = 0.9,\ \mathrm{R}_{0.9} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\lambda = 0,\ \mathrm{R}_0 = \underset{\sim}{\mathbf{E}} \text{ (whole relation; see Chapter 3)}$$

λ-cuts on fuzzy relations obey certain properties, just as λ-cuts on fuzzy sets do (see Eqs. (4.1)), as given in Eqs. (4.2):

1. $(\underset{\sim}{\mathrm{R}} \cup \underset{\sim}{\mathrm{S}})_\lambda = \mathrm{R}_\lambda \cup \mathrm{S}_\lambda$ (4.2a)
2. $(\underset{\sim}{\mathrm{R}} \cap \underset{\sim}{\mathrm{S}})_\lambda = \mathrm{R}_\lambda \cap \mathrm{S}_\lambda$ (4.2b)
3. $(\overline{\underset{\sim}{\mathrm{R}}})_\lambda \neq \overline{\mathrm{R}}_\lambda$ (4.2c)
4. For any $\lambda \leq \alpha, 0 \leq \alpha \leq 1$, then $\mathrm{R}_\alpha \subseteq \mathrm{R}_\lambda$ (4.2d)

DEFUZZIFICATION TO SCALARS

As mentioned in the introduction, there may be situations where the output of a fuzzy process needs to be a single scalar quantity as opposed to a fuzzy set. Defuzzification is the conversion of a fuzzy quantity to a precise quantity, just as fuzzification is the conversion of a precise quantity to a fuzzy quantity. The output of a fuzzy process can be the logical union of two or more fuzzy membership functions defined on the universe of discourse of the output variable. For example, suppose a fuzzy output is comprised of two parts: the first part, $\underset{\sim}{\mathrm{C}}_1$, a trapezoidal shape, shown in Fig. 4.11*a*, and the second part, $\underset{\sim}{\mathrm{C}}_2$, a triangular membership shape, shown in Fig. 4.11*b*. The union of these two membership functions, i.e., $\underset{\sim}{\mathrm{C}} = \underset{\sim}{\mathrm{C}}_1 \cup \underset{\sim}{\mathrm{C}}_2$, involves the max operator, which graphically is the outer envelope of the two shapes shown in Figs. 4.11*a* and *b*; the resulting shape is shown in Fig. 4.11*c*. Of course, a general fuzzy output process can involve many output parts (more than two), and the membership function representing each part of the output can have shapes other than triangles and trapezoids. Further, as Fig. 4.11*a* shows, the membership functions may not always be normal. In general, we can have

$$\underset{\sim}{\mathrm{C}}_k = \bigcup_{i=1}^{k} \underset{\sim}{\mathrm{C}}_i = \underset{\sim}{\mathrm{C}} \tag{4.3}$$

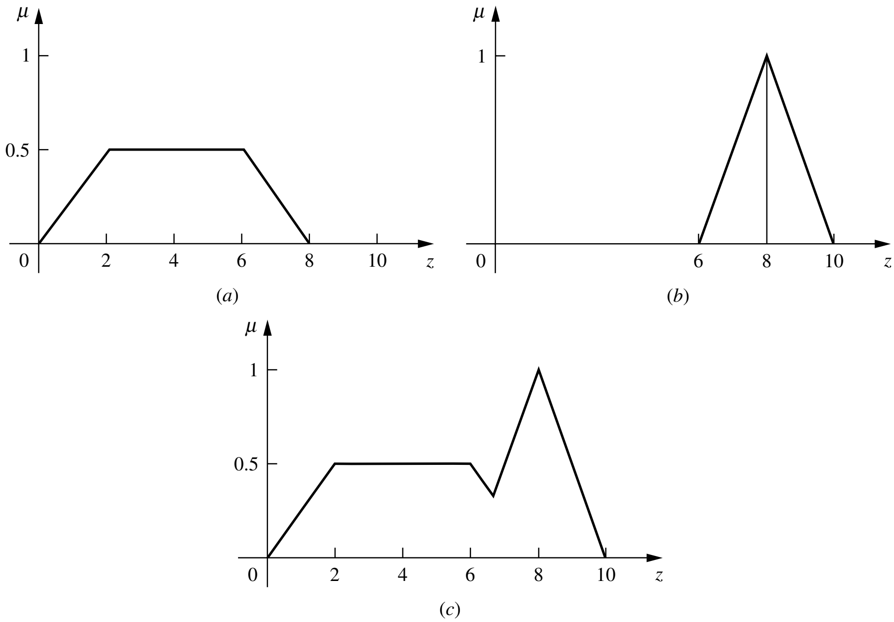

FIGURE 4.11
Typical fuzzy process output: (*a*) first part of fuzzy output; (*b*) second part of fuzzy output; (*c*) union of both parts.

Among the many methods that have been proposed in the literature in recent years, seven are described here for defuzzifying fuzzy output functions (membership functions) [Hellendoorn and Thomas, 1993]. Four of these methods are first summarized and then illustrated in two examples; then the additional three methods are described, then illustrated in two other examples.

1. *Max membership principle*: Also known as the *height* method, this scheme is limited to peaked output functions. This method is given by the algebraic expression

$$\mu_{\underset{\sim}{C}}(z^*) \geq \mu_{\underset{\sim}{C}}(z) \qquad \text{for all } z \in Z \tag{4.4}$$

where z^* is the defuzzified value, and is shown graphically in Fig. 4.12.

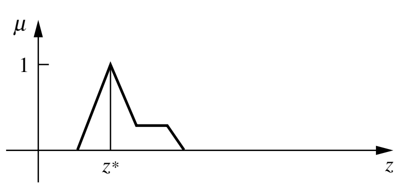

FIGURE 4.12
Max membership defuzzification method.

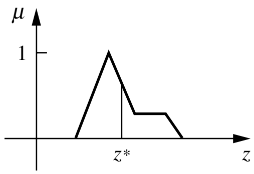

FIGURE 4.13
Centroid defuzzification method.

2. *Centroid method*: This procedure (also called center of area, center of gravity) is the most prevalent and physically appealing of all the defuzzification methods [Sugeno, 1985; Lee, 1990]; it is given by the algebraic expression

$$z^* = \frac{\int \mu_{\underset{\sim}{C}}(z) \cdot z \, \mathrm{d}z}{\int \mu_{\underset{\sim}{C}}(z) \, \mathrm{d}z} \tag{4.5}$$

where $\int$ denotes an algebraic integration. This method is shown in Fig. 4.13.

3. *Weighted average method*: The weighted average method is the most frequently used in fuzzy applications since it is one of the more computationally efficient methods. Unfortunately it is *usually* restricted to symmetrical output membership functions. It is given by the algebraic expression

$$z^* = \frac{\sum \mu_{\underset{\sim}{C}}(\overline{z}) \cdot \overline{z}}{\sum \mu_{\underset{\sim}{C}}(\overline{z})} \tag{4.6}$$

where $\sum$ denotes the algebraic sum and where $\overline{z}$ is the centroid of each symmetric membership function. This method is shown in Fig. 4.14. The weighted average method is formed by weighting each membership function in the output by its respective maximum membership value. As an example, the two functions shown in Fig. 4.14 would result in the following general form for the defuzzified value:

$$z^* = \frac{a(0.5) + b(0.9)}{0.5 + 0.9}$$

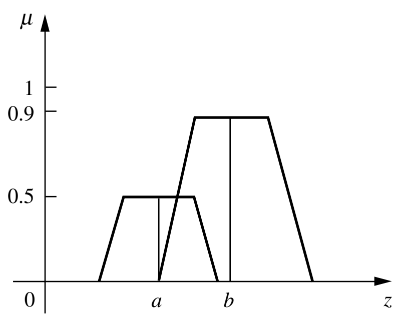

FIGURE 4.14
Weighted average method of defuzzification.

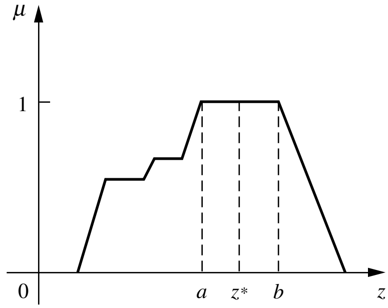

FIGURE 4.15
Mean max membership defuzzification method.

Since the method is limited to symmetrical membership functions, the values a and b are the means (centroids) of their respective shapes.

4. *Mean max membership*: This method (also called middle-of-maxima) is closely related to the first method, except that the locations of the maximum membership can be nonunique (i.e., the maximum membership can be a plateau rather than a single point). This method is given by the expression [Sugeno, 1985; Lee, 1990]

$$z^* = \frac{a+b}{2} \tag{4.7}$$

where a and b are as defined in Fig. 4.15.

Example 4.3. A railroad company intends to lay a new rail line in a particular part of a county. The whole area through which the new line is passing must be purchased for right-of-way considerations. It is surveyed in three stretches, and the data are collected for analysis. The surveyed data for the road are given by the sets, $\underset{\sim}{B}_1$, $\underset{\sim}{B}_2$, and $\underset{\sim}{B}_3$, where the sets are defined on the universe of right-of-way widths, in meters. For the railroad to purchase the land, it must have an assessment of the amount of land to be bought. The three surveys on right-of-way width are ambiguous, however, because some of the land along the proposed railway route is already public domain and will not need to be purchased. Additionally, the original surveys are so old (*circa* 1860) that some ambiguity exists on boundaries and public right-of-way for old utility lines and old roads. The three fuzzy sets, $\underset{\sim}{B}_1$, $\underset{\sim}{B}_2$, and $\underset{\sim}{B}_3$, shown in Figs. 4.16, 4.17, and 4.18, respectively, represent the uncertainty in each survey as to the membership of right-of-way width, in meters, in privately owned land.

We now want to aggregate these three survey results to find the single most nearly representative right-of-way width (z) to allow the railroad to make its initial estimate of the

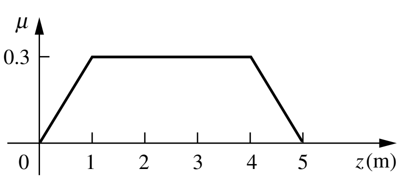

FIGURE 4.16
Fuzzy set $\underset{\sim}{B}_1$: public right-of-way width (z) for survey 1.

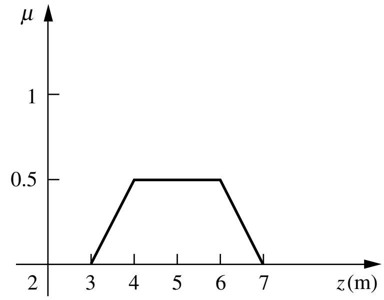

FIGURE 4.17
Fuzzy set $\underset{\sim}{B}_2$: public right-of-way width (z) for survey 2.

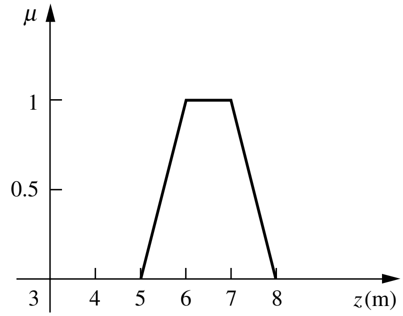

FIGURE 4.18
Fuzzy set $\underset{\sim}{B}_3$: public right-of-way width (z) for survey 3.

right-of-way purchasing cost. Using Eqs. (4.5)–(4.7) and the preceding three fuzzy sets, we want to find z^*.

According to the centroid method, Eq. (4.5), z^* can be found using

$$
\begin{aligned}
z^* &= \frac{\int \mu_{\underset{\sim}{B}}(z) \cdot z\,\mathrm{d}z}{\int \mu_{\underset{\sim}{B}}(z)\,\mathrm{d}z} \\
&= \left[\int_0^1 (0.3z)z\,\mathrm{d}z + \int_1^{3.6} (0.3z)\,\mathrm{d}z + \int_{3.6}^4 \left(\frac{z-3.6}{2}\right) z\,\mathrm{d}z + \int_4^{5.5} (0.5)z\,\mathrm{d}z \right.\\
&\quad \left. + \int_{5.5}^6 (z-5.5)z\,\mathrm{d}z + \int_6^7 z\,\mathrm{d}z + \int_7^8 \left(\frac{7-z}{2}\right) z\,\mathrm{d}z\right] \\
&\quad \div \left[\int_0^1 (0.3z)\,\mathrm{d}z + \int_1^{3.6} (0.3)\,\mathrm{d}z + \int_{3.6}^4 \left(\frac{z-3.6}{2}\right)\mathrm{d}z + \int_4^{5.5} (0.5)\,\mathrm{d}z \right.\\
&\quad \left. + \int_{5.5}^6 \left(\frac{z-5.5}{2}\right)\mathrm{d}z + \int_6^7 \mathrm{d}z + \int_7^8 \left(\frac{7-z}{2}\right)\mathrm{d}z\right] \\
&= 4.9 \text{ meters}
\end{aligned}
$$

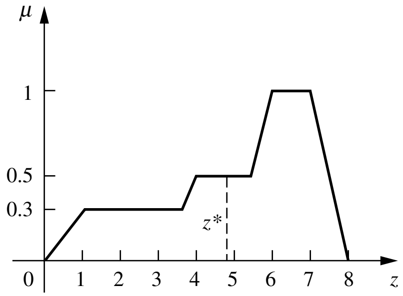

FIGURE 4.19
The centroid method for finding z^*.

where z^* is shown in Fig. 4.19. According to the weighted average method, Eq. (4.6),

$$z^* = \frac{(0.3 \times 2.5) + (0.5 \times 5) + (1 \times 6.5)}{0.3 + 0.5 + 1} = 5.41 \text{ meters}$$

and is shown in Fig. 4.20. According to the mean max membership method, Eq. (4.7), z^* is given by $(6 + 7)/2 = 6.5$ meters, and is shown in Fig. 4.21.

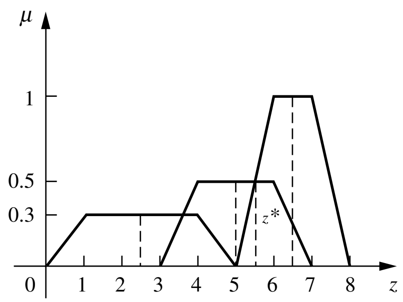

FIGURE 4.20
The weighted average method for finding z^*.

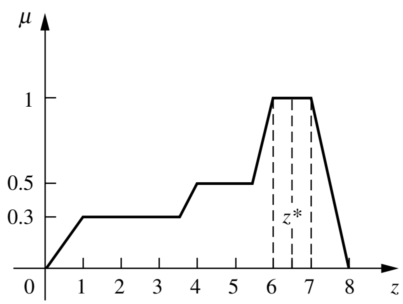

FIGURE 4.21
The mean max membership method for finding z^*.

Example 4.4. Many products, such as tar, petroleum jelly, and petroleum, are extracted from crude oil. In a newly drilled oil well, three sets of oil samples are taken and tested for their viscosity. The results are given in the form of the three fuzzy sets $\underset{\sim}{B}_1$, $\underset{\sim}{B}_2$, and $\underset{\sim}{B}_3$, all defined on a universe of normalized viscosity, as shown in Figs. 4.22–4.24. Using Eqs. (4.4)–(4.6), we want to find the most nearly representative viscosity value for all three oil samples, and hence find z^* for the three fuzzy viscosity sets.

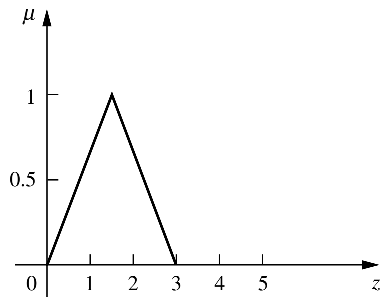

FIGURE 4.22
Membership in viscosity of oil sample 1, $\underset{\sim}{B}_1$.

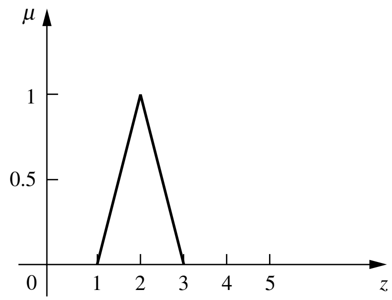

FIGURE 4.23
Membership in viscosity of oil sample 2, $\underset{\sim}{B}_2$.

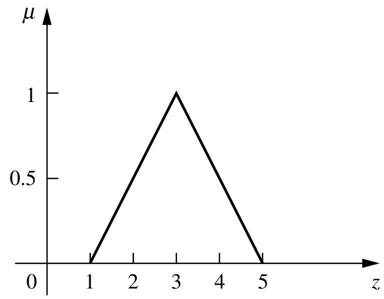

FIGURE 4.24
Membership in viscosity of oil sample 3, $\underset{\sim}{B}_3$.

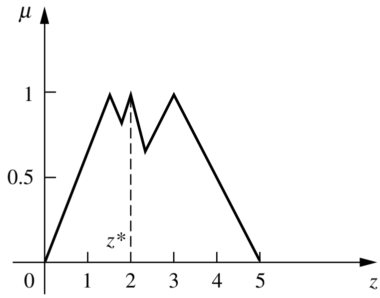

FIGURE 4.25
Logical union of three fuzzy sets $\underset{\sim}{B}_1$, $\underset{\sim}{B}_2$, and $\underset{\sim}{B}_3$.

To find z^* using the centroid method, we first need to find the logical union of the three fuzzy sets. This is shown in Fig. 4.25. Also shown in Fig. 4.25 is the result of the max membership method, Eq. (4.4). For this method, we see that $\mu_{\underset{\sim}{B}}(z^*)$ has three locations where the membership equals unity. This result is ambiguous and, in this case, the selection of the intermediate point is arbitrary, but it is closer to the centroid of the area shown in Fig. 4.25. There could be other compelling reasons to select another value in this case; perhaps max membership is not a good metric for this problem.

According to the centroid method, Eq. (4.5),

$$z^* = \frac{\int \mu_{\underset{\sim}{B}}(z) z\, dz}{\int \mu_{\underset{\sim}{B}}(z)\, dz}$$

$$= \left[\int_0^{1.5} (0.67z) z\, dz + \int_{1.5}^{1.8} (2 - 0.67z) z\, dz + \int_{1.8}^{2} (z - 1) z\, dz + \int_{2}^{2.33} (3 - z) z\, dz \right.$$
$$\left. + \int_{2.33}^{3} (0.5z - 0.5) z\, dz + \int_{3}^{5} (2.5 - 0.5z) z\, dz \right]$$
$$\div \left[\int_0^{1.5} (0.67z)\, dz + \int_{1.5}^{1.8} (2 - 0.67z)\, dz + \int_{1.8}^{2} (z - 1)\, dz + \int_{2}^{2.33} (3 - z)\, dz \right.$$
$$\left. + \int_{2.33}^{3} (0.5z - 0.5)\, dz + \int_{3}^{5} (2.5 - 0.5z)\, dz \right]$$
$$= 2.5 \text{ meters}$$

The centroid value obtained, z^*, is shown in Fig. 4.26.

According to the weighted average method, Eq. (4.6),

$$z^* = \frac{(1 \times 1.5) + (1 \times 2) + (1 \times 3)}{1 + 1 + 1} = 2.25 \text{ meters}$$

and is shown in Fig. 4.27.

Three other popular methods, which are worthy of discussion because of their appearance in some applications, are the center of sums, center of largest area, and first of maxima methods [Hellendoorn and Thomas, 1993]. These methods are now developed.

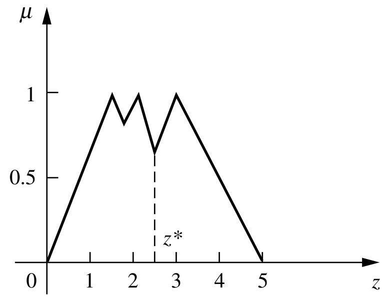

FIGURE 4.26
Centroid value z^* for three fuzzy oil samples.

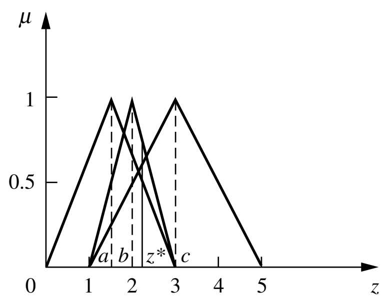

FIGURE 4.27
Weighted average method for z^*.

5. *Center of sums*: This is faster than many defuzzification methods that are presently in use, and the method is not restricted to symmetric membership functions. This process involves the algebraic sum of individual output fuzzy sets, say $\underset{\sim}{C}_1$ and $\underset{\sim}{C}_2$, instead of their union. Two drawbacks to this method are that the intersecting areas are added twice, and the method also involves finding the centroids of the individual membership functions. The defuzzified value z^* is given by the following equation:

$$z^* = \frac{\displaystyle\int_Z \bar{z} \sum_{k=1}^{n} \mu_{\underset{\sim}{C}_k}(z)\,\mathrm{d}z}{\displaystyle\int_z \sum_{k=1}^{n} \mu_{\underset{\sim}{C}_k}(z)\,\mathrm{d}z} \tag{4.8}$$

where the symbol $\bar{z}$ is the distance to the centroid of each of the respective membership functions.

This method is similar to the weighted average method, Eq. (4.6), except in the center of sums method the weights are the areas of the respective membership functions whereas in the weighted average method the weights are individual membership values. Figure 4.28 is an illustration of the center of sums method.

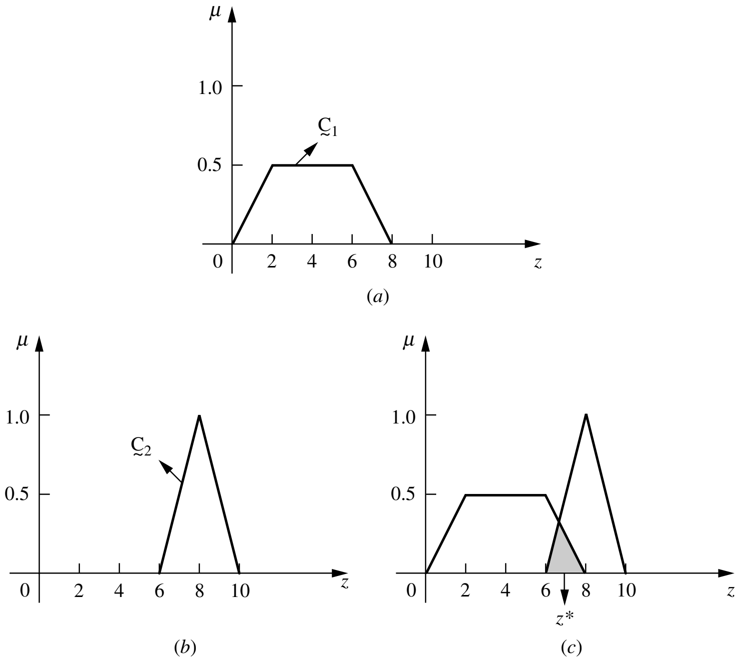

FIGURE 4.28
Center of sums method: (*a*) first membership function; (*b*) second membership function; and (*c*) defuzzification step.

6. *Center of largest area*: If the output fuzzy set has at least two convex subregions, then the center of gravity (i.e., z^* is calculated using the centroid method, Eq. (4.5)) of the convex fuzzy subregion with the largest area is used to obtain the defuzzified value z^* of the output. This is shown graphically in Fig. 4.29, and given algebraically here:

$$z^* = \frac{\int \mu_{\underset{\sim}{C}_m}(z) z \, \mathrm{d}z}{\int \mu_{\underset{\sim}{C}_m}(z) \, \mathrm{d}z} \tag{4.9}$$

   where $\underset{\sim}{C}_m$ is the convex subregion that has the largest area making up $\underset{\sim}{C}_k$. This condition applies in the case when the overall output $\underset{\sim}{C}_k$ is nonconvex; and in the case when $\underset{\sim}{C}_k$ is convex, z^* is the same quantity as determined by the centroid method or the center of largest area method (because then there is only one convex region).
7. *First (or last) of maxima*: This method uses the overall output or union of all individual output fuzzy sets $\underset{\sim}{C}_k$ to determine the smallest value of the domain with maximized membership degree in $\underset{\sim}{C}_k$. The equations for z^* are as follows.

   First, the largest height in the union (denoted $\text{hgt}(\underset{\sim}{C}_k)$) is determined,

$$\text{hgt}(\underset{\sim}{C}_k) = \sup_{z \in Z} \mu_{\underset{\sim}{C}_k}(z) \tag{4.10}$$

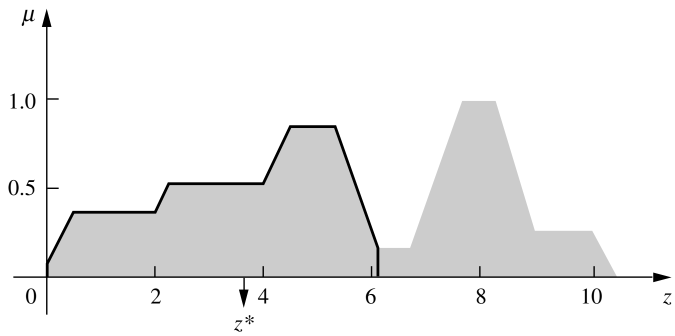

FIGURE 4.29
Center of largest area method (outlined with bold lines), shown for a nonconvex $\underset{\sim}{C}_k$.

Then the first of the maxima is found,

$$z* = \inf_{z \in Z} \left\{ z \in Z \mid \mu_{\underset{\sim}{C}_k}(z) = \text{hgt}(\underset{\sim}{C}_k) \right\} \tag{4.11}$$

An alternative to this method is called the last of maxima, and it is given by

$$z^* = \sup_{z \in Z} \left\{ z \in Z \mid \mu_{\underset{\sim}{C}_k}(z) = \text{hgt}(\underset{\sim}{C}_k) \right\} \tag{4.12}$$

In Eqs. (4.10)–(4.12) the supremum (sup) is the least upper bound and the infimum (inf) is the greatest lower bound. Graphically, this method is shown in Fig. 4.30, where, in the case illustrated in the figure, the first max is also the last max and, because it is a distinct max, is also the mean max. Hence, the methods presented in Eqs. (4.4) (max or height), (4.7) (mean max), (4.11) (first max), and (4.12) (last max) all provide the same defuzzified value, z^*, for the particular situation illustrated in Fig. 4.30.

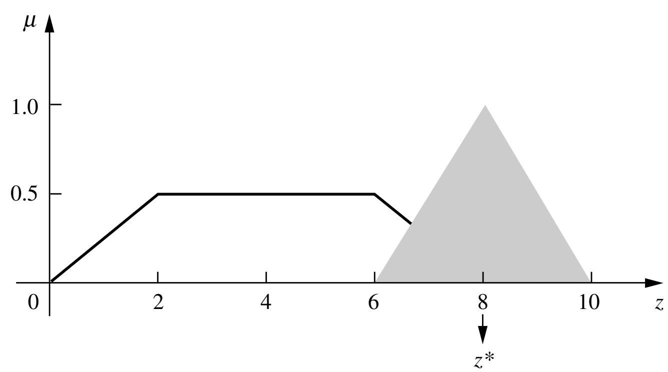

FIGURE 4.30
First of max (and last of max) method.

The problems illustrated in Examples 4.3 and 4.4 are now continued, to illustrate the last three methods presented.

Example 4.5. Continuing with Example 4.3 on the railroad company planning to lay a new rail line, we will calculate the defuzzified values using the (1) center of sums method, (2) center of largest area, and (3) first maxima and last maxima.

According to the center of sums method, Eq. (4.8), z^* will be as follows:

$$z^* = \frac{[2.5 \times 0.5 \times 0.3(3+5) + 5 \times 0.5 \times 0.5(2+4) + 6.5 \times 0.5 \times 1(3+1)]}{[0.5 \times 0.3(3+5) + 0.5 \times 0.5(2+4) + 0.5 \times 1(3+1)]}$$
$$= 5.0 \text{ m}$$

with the result shown in Fig. 4.31. The center of largest area method, Eq. (4.9), provides the same result (i.e., $z^* = 4.9$) as the centroid method, Eq. (4.5), because the complete output fuzzy set is convex, as seen in Fig. 4.32. According to the first of maxima and last of maxima methods, Eqs. (4.11) and (4.12), z^* is shown as z_1^* and z_2^*, respectively, in Fig. 4.33.

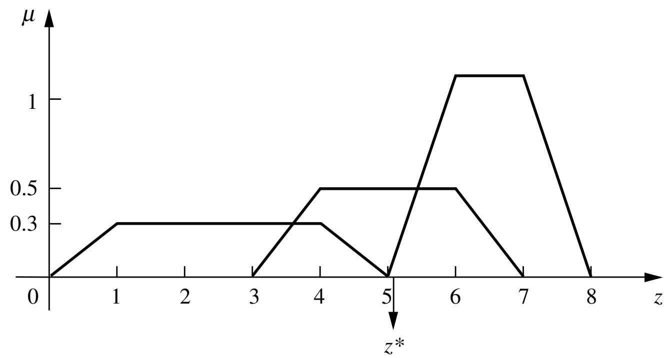

FIGURE 4.31
Center of sums result for Example 4.5.

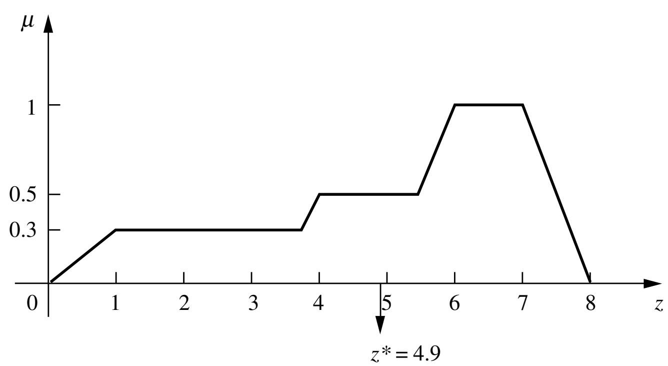

FIGURE 4.32
Output fuzzy set for Example 4.5 is convex.

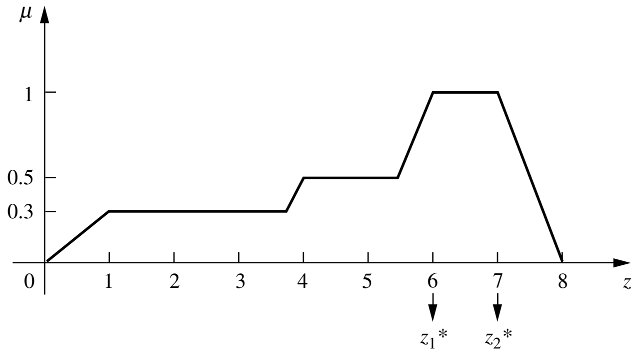

FIGURE 4.33
First of maxima solution ($z_1^* = 6$) and last of maxima solution ($z_2^* = 7$).

Example 4.6. Continuing with Example 4.4 on the crude oil problem, the center of sums method, Eq. (4.8), produces a defuzzified value for z^* of

$$z^* = \frac{(0.5 \times 3 \times 1 \times 1.5 + 0.5 \times 2 \times 1 \times 2 + 0.5 \times 4 \times 1 \times 3)}{(0.5 \times 3 \times 1 + 0.5 \times 2 \times 1 + 0.5 \times 4 \times 1)} = 2.3 \text{ m}$$

which is shown in Fig. 4.34. In the center of largest area method we first determine the areas of the three individual convex fuzzy output sets, as seen in Fig. 4.35. These areas are 1.02, 0.46, and 1.56 square units, respectively. Among them, the third area is largest, so the centroid of that area will be the center of the largest area. The defuzzified value is calculated to be $z^* = 3.3$:

$$z^* = \frac{\left[\left(\frac{0.67}{2} + 2.33\right)[0.5 \times 0.67(1 + 0.67)]\right] + 3.66(0.5 \times 2 \times 1)}{[0.5 \times 0.67(1 + 0.67)] + (0.5 \times 2 \times 1)} = 3.3 \text{ m}$$

Finally, one can see graphically in Fig. 4.36 that the first of maxima and last of maxima, Eqs. (4.11)–(4.12), give different values for z^*, namely, $z^* = 1.5$ and 3.0, respectively.

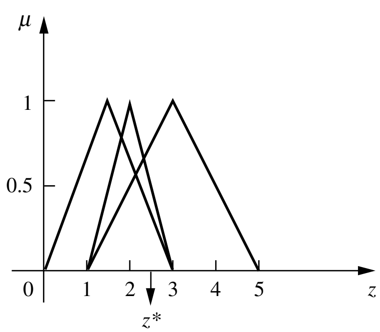

FIGURE 4.34
Center of sums solution for Example 4.6.

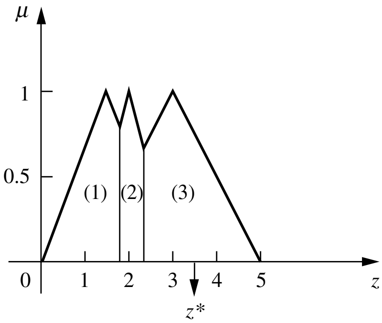

FIGURE 4.35
Center of largest area method for Example 4.6.

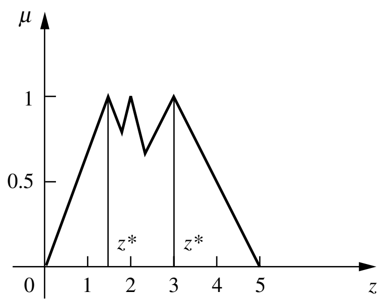

FIGURE 4.36
First of maxima gives $z^* = 1.5$ and last of maxima gives $z^* = 3$.

SUMMARY

This chapter has introduced the various features and forms of a membership function and the idea of fuzzyifying scalar quantities to make them fuzzy sets. The primary focus of the chapter, however, has been to explain the process of converting from fuzzy membership functions to crisp formats – a process called defuzzification. Defuzzification is necessary because, for example, we cannot instruct the voltage going into a machine to increase ''slightly,'' even if this instruction comes from a fuzzy controller – we must alter its voltage by a specific amount. Defuzzification is a natural and necessary process. In fact, there is an analogous form of defuzzification in mathematics where we solve a complicated problem in the complex plane, find the real and imaginary parts of the solution, then *decomplexify* the imaginary solution back to the real numbers space [Bezdek, 1993]. There are numerous other methods for defuzzification that have not been presented here. A review of the literature will provide the details on some of these [see, for example, Filev and Yager, 1991; Yager and Filev, 1993].

A natural question to ask is: Of the seven defuzzification methods presented, which is the best? One obvious answer to the question is that it is context- or problem-dependent.

To answer this question in more depth, Hellendoorn and Thomas [1993] have specified five criteria against which to measure the methods. These criteria will be repeated here for the benefit of the reader who also ponders the question just given in terms of the advantages and disadvantages of the various methods. The first criterion is *continuity*. A small change in the input of a fuzzy process should not produce a large change in the output. Second, a criterion known as *disambiguity* simply points out that a defuzzification method should always result in a unique value for z^*, i.e., no ambiguity in the defuzzified value. This criterion is not satisfied by the center of largest area method, Eq. (4.9), because, as seen in Fig. 4.31, when the largest membership functions have equal area, there is ambiguity in selecting a z^*. The third criterion is called *plausibility*. To be plausible, z^* should lie approximately in the middle of the support region of $\underset{\sim}{C}_k$ and have a high degree of membership in $\underset{\sim}{C}_k$. The centroid method, Eq. (4.5), does not exhibit plausibility in the situation illustrated in Fig. 4.31 because, although z^* lies in the middle of the support of $\underset{\sim}{C}_k$, it does not have a high degree of membership (also seen in the darkened area of Fig. 4.28*c*). The fourth criterion is that of *computational simplicity*, which suggests that the more time consuming a method is, the less value it should have in a computation system. The height method, Eq. (4.4), the mean max method, Eq. (4.7), and the first of maxima method are faster than the centroid, Eq. (4.5), or center of sum, Eq. (4.8), methods, for example. The fifth criterion is called the *weighting method*, which weights the output fuzzy sets. This criterion constitutes the difference between the centroid method, Eq. (4.5), the weighted average method, Eq. (4.6), and center of sum methods, Eq. (4.8). The problem with the fifth criterion is that it is problem-dependent, as there is little by which to judge the best weighting method; the weighted average method involves less computation than the center of sums, but that attribute falls under the fourth criterion, computational simplicity.

As with many issues in fuzzy logic, the method of defuzzification should be assessed in terms of the goodness of the answer in the context of the data available. Other methods are available that purport to be superior to the simple methods presented here [Hellendoorn and Thomas, 1993].

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PROBLEMS

4.1. Two fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$, both defined on X, are as follows:

$\mu(x_i)$	x_1	x_2	x_3	x_4	x_5	x_6
$\underset{\sim}{A}$	0.1	0.6	0.8	0.9	0.7	0.1
$\underset{\sim}{B}$	0.9	0.7	0.5	0.2	0.1	0

Express the following λ-cut sets using Zadeh's notation:

(a) $(\overline{\underset{\sim}{A}})_{0.7}$
(b) $(\underset{\sim}{B})_{0.4}$
(c) $(\underset{\sim}{A} \cup \underset{\sim}{B})_{0.7}$
(d) $(\underset{\sim}{A} \cap \underset{\sim}{B})_{0.6}$
(e) $(\underset{\sim}{A} \cup \overline{\underset{\sim}{A}})_{0.7}$
(f) $(\underset{\sim}{B} \cap \overline{\underset{\sim}{B}})_{0.5}$
(g) $(\overline{\underset{\sim}{A} \cap \underset{\sim}{B}})_{0.7}$
(h) $(\overline{\underset{\sim}{A}} \cup \overline{\underset{\sim}{B}})_{0.7}$

4.2. [Klir and Folger, 1988] Show that all λ-cuts of any fuzzy set $\underset{\sim}{A}$ defined in R^n space ($n \geq 1$) are convex if and only if

$$\mu_{\underset{\sim}{A}}[\lambda r + (1 - \lambda)s] \geq \min[\mu_{\underset{\sim}{A}}(r), \mu_{\underset{\sim}{A}}(s)]$$

for all $r, s \in R^n$, and all $\lambda \in [0, 1]$.

4.3. The fuzzy sets $\underset{\sim}{A}$, $\underset{\sim}{B}$, and $\underset{\sim}{C}$ are all defined on the universe $X = [0, 5]$ with the following membership functions:

$$\mu_{\underset{\sim}{A}}(x) = \frac{1}{1 + 5(x - 5)^2} \qquad \mu_{\underset{\sim}{B}}(x) = 2^{-x} \qquad \mu_{\underset{\sim}{C}}(x) = \frac{2x}{x + 5}$$

(a) Sketch the membership functions.
(b) Define the intervals along the x axis corresponding to the λ-cut sets for each of the fuzzy sets $\underset{\sim}{A}$, $\underset{\sim}{B}$, and $\underset{\sim}{C}$ for the following values of λ:
(i) $\lambda = 0.2$
(ii) $\lambda = 0.4$
(iii) $\lambda = 0.7$
(iv) $\lambda = 0.9$
(v) $\lambda = 1.0$

4.4. Determine the crisp λ-cut relations for $\lambda = 0.1j$, for $j = 0,1,\ldots,10$, for the following fuzzy relation matrix $\underset{\sim}{R}$:

$$\underset{\sim}{R} = \begin{bmatrix} 0.2 & 0.7 & 0.8 & 1 \\ 1 & 0.9 & 0.5 & 0.1 \\ 0 & 0.8 & 1 & 0.6 \\ 0.2 & 0.4 & 1 & 0.3 \end{bmatrix}$$

4.5. For the fuzzy relation $\underset{\sim}{R}_4$ in Example 3.11 find the λ-cut relations for the following values of λ:
(a) $\lambda = 0^+$
(b) $\lambda = 0.1$
(c) $\lambda = 0.4$
(d) $\lambda = 0.7$

4.6. For the fuzzy relation $\underset{\sim}{R}$ in Problem 3.9(a) sketch (in 3D) the λ-cut relations for the following values of λ:

(*a*) $\lambda = 0^+$
(*b*) $\lambda = 0.3$
(*c*) $\lambda = 0.5$
(*d*) $\lambda = 0.9$
(*e*) $\lambda = 1$

4.7. Show that any λ-cut relation (for $\lambda > 0$) of a fuzzy tolerance relation results in a crisp tolerance relation.

4.8. Show that any λ-cut relation (for $\lambda > 0$) of a fuzzy equivalence relation results in a crisp equivalence relation.

4.9. In metallurgy materials are made with mixtures of various metals and other elements to achieve certain desirable properties. In a particular preparation of steel, three elements, namely iron, manganese, and carbon, are mixed in two different proportions. The samples obtained from these two different proportions are placed on a normalized scale, as shown in Fig. P4.9 and are represented as fuzzy sets $\underset{\sim}{A}_1$ and $\underset{\sim}{A}_2$. You are interested in finding some sort of ''average'' steel proportion. For the logical union of the membership functions shown we want to find the defuzzified quantity. For each of the seven methods presented in this chapter assess (*a*) whether each is applicable and, if so, (*b*) calculate the defuzzified value, z^*.

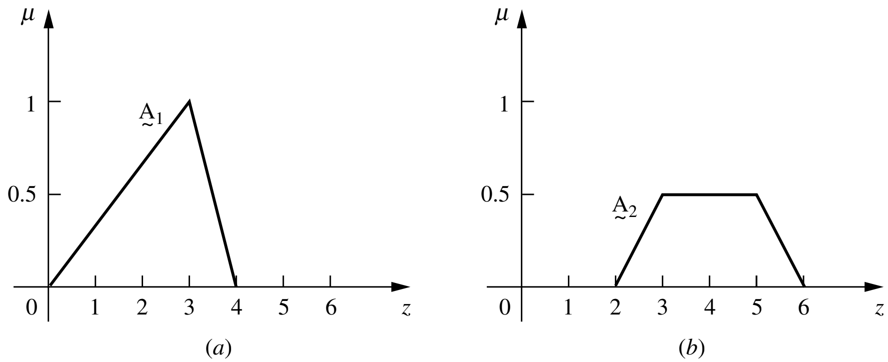

FIGURE P4.9

4.10. Two companies bid for a contract. A committee has to review the estimates of those companies and give reports to its chairperson. The reviewed reports are evaluated on a nondimensional scale and assigned a weighted score that is represented by a fuzzy membership function, as illustrated by the two fuzzy sets, $\underset{\sim}{B}_1$ and $\underset{\sim}{B}_2$, in Fig. P4.10. The chairperson is interested in the lowest bid, as well as a metric to measure the combined ''best'' score. For the logical union of the membership functions shown we want to find the defuzzified quantity. For each of the seven methods presented in this chapter assess (*a*) whether each is applicable and, if so, (*b*) calculate the defuzzified value, z^*.

4.11. A landfill is the cheapest method of solid waste treatment and disposal. Once disposed into a landfill, solid waste can be a major source of energy due to its potential to produce methane. However, all the solid waste disposed cannot generate methane at the same rate and in the same quantities. Based on its biodegradability, solid waste is classified into three distinct groups, namely: rapidly biodegradable, moderately biodegradable, and slowly biodegradable. Design of a landfill gas extraction system is based on gas production through the first two groups; both have different gas production patterns. The data collected from experiments and experiences are presented by the sets $\underset{\sim}{A}_1$ and $\underset{\sim}{A}_2$ as shown in Fig. P4.11, where $\underset{\sim}{A}_1$ and $\underset{\sim}{A}_2$ are defined as the fuzzy

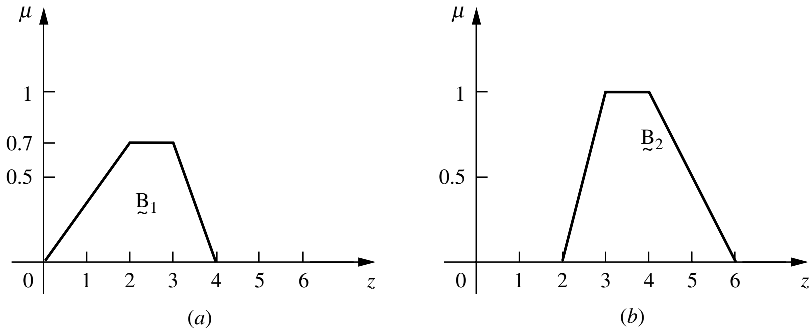

FIGURE P4.10

sets rapidly biodegradable and slowly biodegradable, respectively, in units of years. In order to properly design the gas extraction system we need a single representative gas production value. For the logical union of the membership functions shown we want to find the defuzzified quantity. For each of the seven methods presented in this chapter assess (*a*) whether each is applicable and, if so, (*b*) calculate the defuzzified value, z^*.

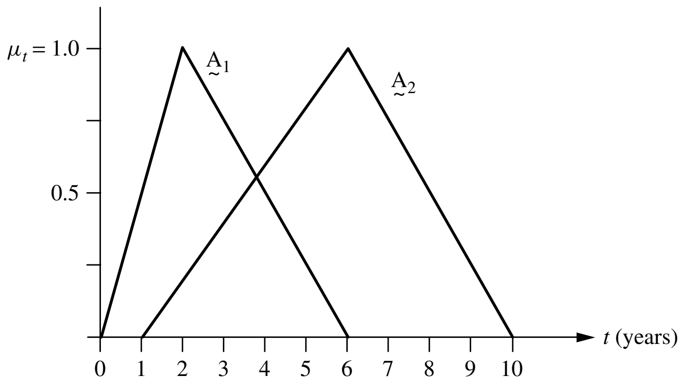

FIGURE P4.11

4.12. Uniaxial compressive strength is easily performed on cylindrical or prismatic ice samples and can vary with strain rate, temperature, porosity, grain orientation, and grain size ratio. While strain rate and temperature can be controlled easily, the other variables cannot. This lack of control yields an uncertainty in the uniaxial test results.

A test was conducted on each type of sample at a constant strain rate of 10^{-4} s^{-1}, and a temperature of $-5°C$. Upon inspection of the results the exact yield point could not be determined; however, there was enough information to form fuzzy sets for the failure of the cylindrical and prismatic samples $\underset{\sim}{A}$ and $\underset{\sim}{B}$, respectively, as shown in Fig. P4.12. Once the union of $\underset{\sim}{A}$ and $\underset{\sim}{B}$ has been identified (the universe of compressive strengths, megapascals $N/m^2 \times 10^6$) we can obtain a defuzzified value for the yield strength of this ice under a compressive axial load. For each of the seven methods presented in this chapter assess (*a*) whether each is applicable and, if so, (*b*) calculate the defuzzified value, z^*.

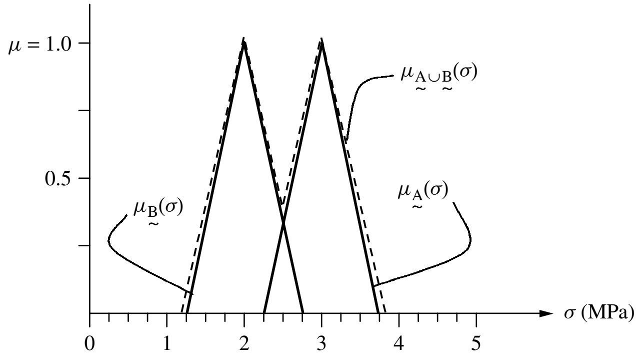

FIGURE P4.12

4.13. In the field of heat exchanger network (HEN) synthesis, a chemical plant can have two different sets of independent HENs. The term ''optimum cost'' is considered fuzzy, because for the design and construction of the HENs we have to consider other parameters in addition to the capital cost. The membership function of fuzzy sets HEN1 and HEN2 is shown in Figs.P4.13(a) and P4.13(b), respectively.

We wish to determine the optimum capital cost of a project to optimize the plant using both independent networks (HEN1 and HEN2); hence, the logical union of their membership functions, as shown in Fig. P4.13(c). For each of the seven methods presented in this chapter assess (*a*) whether each is applicable and, if so, (*b*) calculate the defuzzified value, z^*.

4.14. In reactor design, it is often best to simplify a reactor by assuming ideal conditions. For a continuous stirred tank reactor (CSTR), the concentration inside the reactor is the same as the concentration of the effluent stream. In a plug flow reactor (PFR), the concentrations of the inlet and outlet streams are different as the concentrations of the reactants change along the length of the tube. For a fluidized bed in which catalyst is removed from the top of the reactor, there exists both characteristics of a CSTR and PFR. The difference between inside reactor concentration (C_i) and effluent concentration (C_e) gives the membership of either CSTR or PFR, as seen in Fig. P4.14.

Find the difference in concentration that represents the optimum design, i.e., find the most representative value for the union of PFR and CSTR. For each of the seven methods presented in this chapter assess (*a*) whether each is applicable and, if so, (*b*) calculate the defuzzified value, z^*.

4.15. Often in chemical processing plants there will be more than one type of instrumentation measuring the same variable at the same instance during the process. Due to the nature of measurements they are almost never exact, and hence can be represented as a fuzzy set. Due to the differences in instrumentation the measurements will usually not be the same. Take for example two types of temperature sensors, namely a thermocouple (TC) and an RTD (Resistance Temperature Detector) measuring the same stream temperature. The membership function of the two types of temperature sensors may look as in Fig. P4.15.

When an operator who prefers one measuring device ends his or her shift, and then is replaced by another operator with a different preference in measuring device, there may be a problem in determining the actual value of a variable. To avoid this problem it was decided to plot the membership functions of the two types of sensors, take their union, and employ defuzzification to select one temperature for the operator to use. To find this temperature, for each of the seven methods presented in this chapter, assess (*a*) whether each is applicable and, if so, (*b*) calculate the defuzzified value, z^*.

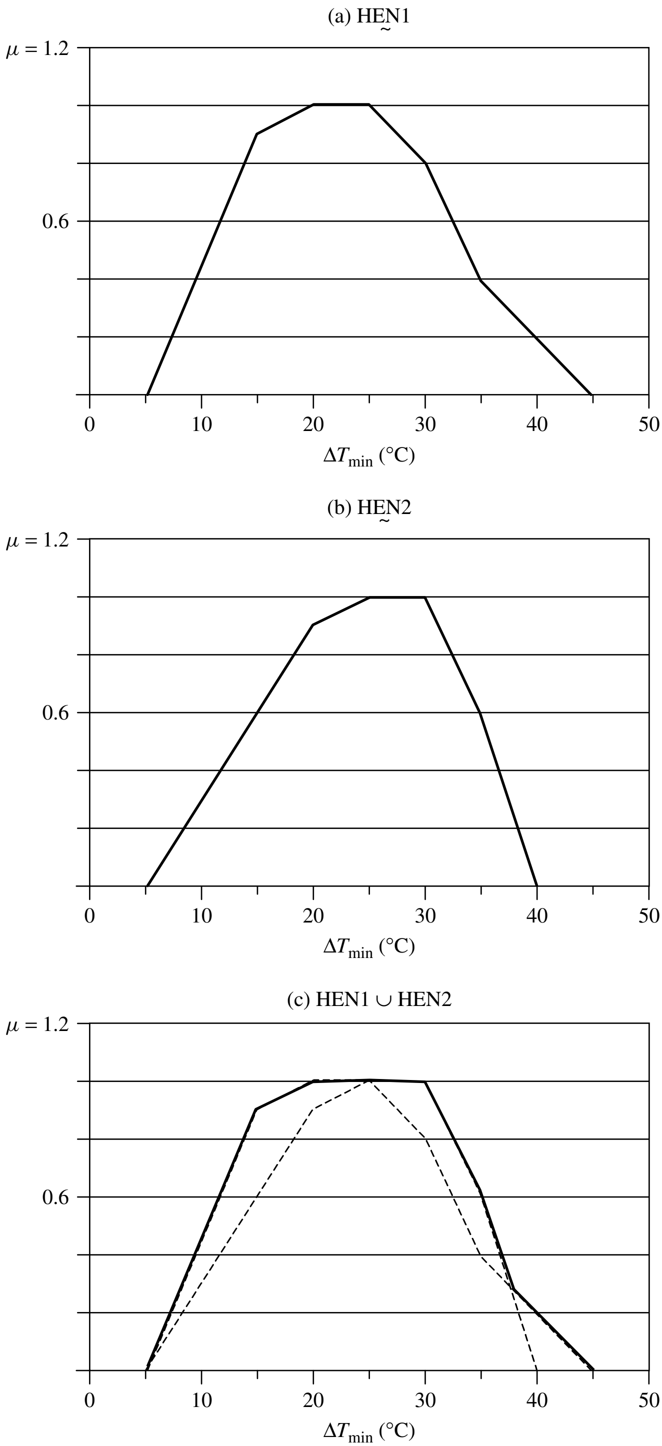
(a) HEN1
μ = 1.2
0.6
0
10
20
30
40
50
ΔT_min (°C)
(b) HEN2
μ = 1.2
0.6
0
10
20
30
40
50
ΔT_min (°C)
(c) HEN1 ∪ HEN2
μ = 1.2
0.6
0
10
20
30
40
50
ΔT_min (°C)

FIGURE P4.13

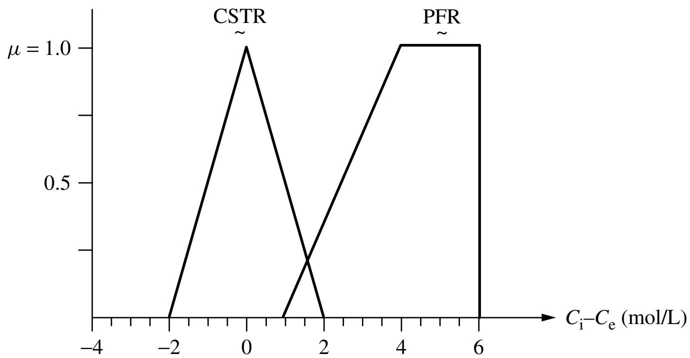

FIGURE P4.14

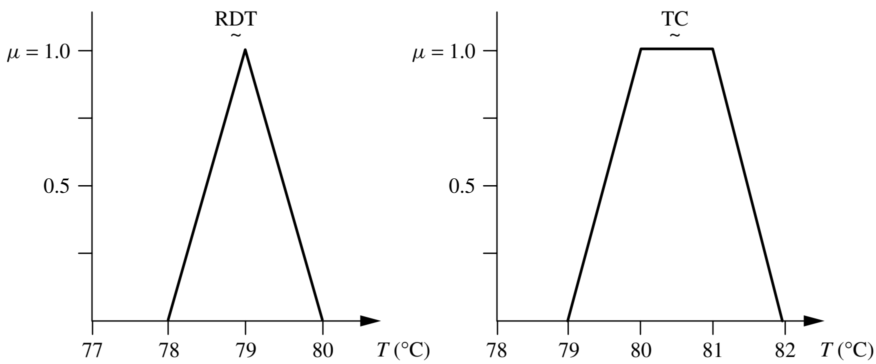

FIGURE P4.15

CHAPTER 5

LOGIC AND FUZZY SYSTEMS

PART I LOGIC

> *''I know what you're thinking about,'' said Tweedledum; ''but it isn't so, nohow.'' ''Contrariwise,'' continued Tweedledee, ''if it was so, it might be; and if it were so, it would be; but as it isn't, it ain't. That's logic.''*
>
> ***Lewis Carroll***
> *Through the Looking Glass,* 1871

Logic is but a small part of the human capacity to reason. Logic can be a means to compel us to infer correct answers, but it cannot by itself be responsible for our creativity or for our ability to remember. In other words, logic can assist us in organizing words to make clear sentences, but it cannot help us determine what sentences to use in various contexts. Consider the passage above from the nineteenth-century mathematician Lewis Carroll in his classic *Through the Looking Glass*. How many of us can see the logical context in the discourse of these fictional characters? Logic for humans is a way quantitatively to develop a reasoning process that can be replicated and manipulated with mathematical precepts. The interest in logic is the study of truth in logical propositions; in classical logic this truth is binary – a proposition is either true or false.

From this perspective, fuzzy logic is a method to formalize the human capacity of imprecise reasoning, or – later in this chapter – approximate reasoning. Such reasoning represents the human ability to reason approximately and judge under uncertainty. In fuzzy logic all truths are partial or approximate. In this sense this reasoning has also been termed interpolative reasoning, where the process of interpolating between the binary extremes of true and false is represented by the ability of fuzzy logic to encapsulate partial truths.

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